Global Sensitivity Indices for Nonlinear Mathematical Models. Review

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Abstract: This is a review of global sensitivity indices that were introduced in I.M. Sobol' (1990). These indices allow to analyze numerically the structure of a nonlinear function defined analytically or by a "black box". As an example the Brownian bridge is considered and an example of the application of global sensitivity indices in finance is presented.

Keywords: Monte Carlo, Quasi-Monte Carlo, Global sensitivity analysis, Brownian bridge.

1 What is Global Sensitivity Analysis

Consider the mathematical model described by a function

$$u = f(x),$$

where the input $x = (x_1, \ldots, x_n)$ is defined in a certain region G, and the output u is a real value. Traditional sensitivity analysis that can be called local, is applied to a specified solution, say $u^* = f(x^*)$. The sensitivity of u^* with respect to the input can be measured using the derivatives

$$(\partial f/\partial x_i)_{x=x^*}.$$

In the global sensitivity approach individual solutions are not considered. The function $f(x)$ in G is studied so that the influence of different variables and their subsets, the structure of $f(x)$ and possible approximations, etc can be analyzed A. Saltelli, K. Chan and M. Scott (2000) .

2 ANOVA-Decomposition

We shall consider square integrable functions $f(x)$, $x = (x_1, \ldots, x_n)$, defined in the unit hypercube $0 \le x_1 \le 1, \ldots, 0 \le x_n \le 1$. In the following text integrals written without limits of integration are from 0 to 1 in each variable.

Definition. The representation of $f(x)$ in a form

$$f(x) = f_0 + \sum_{s=1}^{n} \sum_{i_1 < \cdots < i_s} f_{i_1 \ldots i_s}\left(x_{i_1}, \ldots, x_{i_s}\right) \tag{1}$$

is called ANOVA-decomposition if

$$f_0 = \int f(x)\, dx, \tag{2}$$

and

$$\int_0^1 f_{i_1\dots i_s}\,dx_{i_p} = 0 \quad \text{for} \quad 1 \le p \le s. \tag{3}$$

Here $1 \le i_1 < i_2 < \dots < i_s \le n, \quad 1 \le s \le n$.

The word ANOVA comes from Analysis of Variances. The explicit form of (1) is

$$f(x) = f_0 + \sum_i f_i(x_i) + \sum_{i<j} f_{ij}(x_i, x_j) + \dots + f_{12\dots n}(x_1, x_2, \dots, x_n).$$

One can easily prove that conditions (2) and (3) define uniquely all the terms in (1). Indeed, integrating (1) over all variables except x_i we obtain

$$\int f(x) \prod_{p\ne i} dx_p = f_0 + f_i(x_i).$$

Thus all one-dimensional terms $f_i(x_i)$ are defined. To define the two-dimensional terms $f_{ij}(x_i, x_j)$ we integrate (1) over all variables except x_i and x_j:

$$\int f(x) \prod_{p\ne ij} dx_p = f_0 + f_i(x_i) + f_j(x_j) + f_{ij}(x_i, x_j).$$

And so on. The last term $f_{12\dots n}(x_1, x_2, \dots, x_n)$ is defined by (1).

An important property of (1) is the orthogonality of its terms:

$$\int f_{i_1\dots i_s} f_{k_1\dots k_l}\,dx = 0,$$

if $(i_1, \dots, i_s) \not\equiv (k_1, \dots, k_l)$. This is a direct consequence of (3).

Variances

Constants

$$D_{i_1\dots i_s} = \int f^2_{i_1\dots i_s}\,dx_{i_1}\dots dx_{i_s}$$

are called variances.

$$D = \int f^2(x)\,dx - f_0^2$$

is the total variance.

If x were a random point uniformly distributed in the hypercube, these constants would be real variances.

Squaring (1) and integrating over the hypercube we obtain the relation

$$D = \sum_{s=1}^{n} \sum_{i_1<\dots<i_s} D_{i_1\dots i_s}. \tag{4}$$

The variance $D_{i_1\dots i_s}$ shows the variability of $f_{i_1\dots i_s}$. For piecewise continuous $f_{i_1\dots i_s}$ one can assert that $D_{i_1\dots i_s} = 0$ if and only if $f_{i_1\dots i_s}(x_{i_1}, \dots x_{i_s}) \equiv 0$.

For a more or less complex function $f(x)$, it is impossible to find multidimensional terms of (1). On the contrary, the variances $D_{i_1\dots i_s}$ can be numerically estimated directly from values of $f(x)$.

3 Global Sensitivity Indices

Definition. A global sensitivity index is the ratio of variances

$$S_{i_1\dots i_s} = D_{i_1\dots i_s}/D. \tag{5}$$

It follows from (4) that

$$\sum_{s=1}^{n} \sum_{i_1<\dots<i_s} S_{i_1\dots i_s} = 1. \tag{6}$$

Clearly all sensitivity indices are nonnegative, an index $S_{i_1\dots i_s} = 0$ if and only if $f_{i_1\dots i_s} \equiv 0$.

The following assertion is more or less evident: the function $f(x)$ is a sum of one-dimensional functions if and only if

$$\sum_{i=1}^{n} S_i = 1. \tag{7}$$

One-dimensional sensitivity indices S_i were used in some papers for ranking of the input variables x_i. However, a more detailed analysis requires the use of total sensitivity indices that will be introduced in the next section.

4 Global Sensitivity Indices for Subsets of Variables

Consider an arbitrary subset of variables $x_{k_1}, \dots, x_{k_m}$, where $1 \le k_1 < k_2 < \dots < k_m \le n$ and $1 \le m \le n-1$. We will denote it by one letter $y = (x_{k_1}, \dots, x_{k_m})$, and let z be the set of $n-m$ complementary variables; so that $x = (y, z)$. The set of indices $k_1, \dots, k_m$ will be denoted by K.

Two types of sensitivity indices for the set y are introduced:

Definitions.

$$S_y = \sum S_{i_1,\dots,i_s},$$

where the sum is extended over all sets $i_1, \dots, i_s \in K$;

$$S_y^{t_0 t} = \sum S_{i_1,\dots,i_s},$$

where the sum is extended over all sets $i_1, \dots, i_s$ with at least one index $i_p \in K$; clearly, $0 \le S_y \le S_y^{t_0 t} \le 1$.

The first of the two definitions can be applied for defining S_z. Then $S_y^{t_0 t} = 1 - S_z$ and similarly $S_z^{t_0 t} = 1 - S_y$.

An equivalent approach is to introduce a mixed sensitivity index $S_{y,z} = 1 - S_z - S_y$. Then $S_y^{t_0 t} = S_y + S_{y,z}$, $S_z^{t_0 t} = S_z + S_{y,z}$.

The most informative are the extreme cases:

A) $S_y = S_y^{t_0 t} = 0$ if and only if the function $f(x)$ does not depend on y.
B) $S_y = S_y^{t_0 t} = 1$ if and only if the function $f(x)$ does not depend on z; (the function $f(x)$ is assumed to be piecewise continuous).

If the set y consists of one variable $y = (x_i)$ then $S_y = S_i$ while $S_y^{t_0 t} = S_i^{t_0 t}$ is the sum of all $S_{i_1,\dots,i_s}$ that contain $i_p = i$.

Example.

Let $f = f(x_1, x_2, x_3)$.

If $y = (x_1)$ then $S_y = S_1$, $S_y^{t_0 t} = S_1 + S_{12} + S_{13} + S_{123}$.

If $y = (x_1, x_2)$ then $z = (x_3)$. Clearly, $S_y = S_1 + S_2 + S_{12}$, $S_z = S_3$, $S_y^{t_0 t} = S_1 + S_2 + S_{12} + S_{13} + S_{23} + S_{123} = 1 - S_3$.

A major advantage of the theory is the somewhat unexpected fact that it is unnecessary to compute sums in the definitions of S_y and $S_y^{t_0 t}$: both these quantities (or more accurately speaking, the corresponding variances D_y and $D_y^{t_0 t}$) can be computed directly from values of $f(x)$ at specially selected random or quasi-random points.

5 Integral Representations for D_y and $D_y^{t_0 t}$

Denote by D_y and $D_y^{t_0 t}$ sums of $D_{i_1 \dots i_s}$ that correspond to the sums in the definitions of S_y and $S_y^{t_0 t}$. Then

$$S_y = \frac{D_y}{D}, \qquad S_y^{t_0 t} = \frac{D_y^{t_0 t}}{D}.$$

Let x and x' be independent variables defined in the same hypercube (or consider the product of two hypercubes). Similarly to $x = (y, z)$ we will write $x' = (y', z')$.

Theorem 1.

$$D_y = \int f(x) f(y, z')\, dx dz' - f_0^2.$$

Proof. The integral on the right hand side can be transformed:

$$\int dy \int f(y, z)\, dz \int f(y, z')\, dz' = \int dy \left[\int f(y, z)\, dz \right]^2.$$

Substituting (1) into the inner integral and integrating over dz we retain only terms depending on y and f_0. They are squared and integrated over dy.

Thus, we obtain $D_y + f_0^2$.

Theorem 2.

$$D_y^{t_0 t} = \frac{1}{2} \int \left[f(x) - f(y', z) \right]^2 dx dy'.$$

Proof. The expression on the right hand side is equal to

$$\int f^2(x)\, dx - \int f(x) f(y', z)\, dx dy'.$$

According to Theorem 1, this is equal to

$$\int f^2(x)\, dx - \left(D_z + f_0^2 \right) = D - D_z$$

6 A Monte Carlo Algorithm

For the k-th trial we generate two m-dimensional random points η_k and η'_k and two $(n-m)$-dimensional random points ζ_k and ζ'_k. Then we compute the function $f(y, z)$ at three points: $f(\eta_k, \zeta_k)$, $f(\eta_k, \zeta'_k)$ and $f(\eta'_k, \zeta_k)$.

Four estimators are computed: $\varphi_k = f(\eta_k, \zeta_k)$, φ_k^2, $\psi_k = \varphi_k f(\eta_k, \zeta'_k)$ and $\chi_k = \frac{1}{2} \left[\varphi_k - f(\eta'_k, \zeta_k) \right]^2$.

After N independent trials at $N \to \infty$

$$\frac{1}{N} \sum_{k=1}^{N} \varphi_k \xrightarrow{P} f_0,$$

$$\frac{1}{N} \sum_{k=1}^{N} \varphi_k^2 \xrightarrow{P} D + f_0^2,$$

$$\frac{1}{N} \sum_{k=1}^{N} \psi_k \xrightarrow{P} D_y + f_0^2,$$

$$\frac{1}{N} \sum_{k=1}^{N} \chi_k \xrightarrow{P} D_y^{t_0 t}.$$

A quasi-Monte Carlo estimation of f_0, D, D_y and $D_y^{t_0 t}$ is also possible. For the trial number k we select one $2n$-dimensional quasi-random point $Q_k = \left(q_1^k, \dots, q_{2n}^k \right)$ and define $\eta_k = \left(q_1^k, \dots, q_m^k \right)$, $\zeta_k = \left(q_{m+1}^k, \dots, q_n^k \right)$, $\eta'_k = \left(q_{n+1}^k, \dots, q_{m+n}^k \right)$, $\zeta'_k = \left(q_{n+m+1}^k, \dots, q_{2n}^k \right)$.

More information on the computation algorithms can be found in I.M. Sobol' (2001).

7 Low Dimensional Approximations of $f(x)$

According to H. Rabitz, O.F. Alis, J. Shorter and K. Shim (1999) very often in mathematical models $f(x)$ low order interactions of input variables have the main impact upon the output. In such cases a low dimensional approximation $f(x) \approx h_L(x)$, $L \ll n$, where

$$h_L(x) = f_0 + \sum_{s=1}^{L} \sum_{i_1 < \dots < i_s} f_{i_1 \dots i_s}(x_{i_1}, \dots, x_{i_s})$$

can be rather efficient. The construction of such approximations is discussed in H. Rabitz, O.F. Alis, J. Shorter and K. Shim (1999) and I.M. Sobol' (2003).

We will use the scaled L_2 distance for measuring the error of an approximation $f(x) \approx h(x)$:

$$\delta(f, h) = \frac{1}{D} \int \left[f(x) - h(x) \right]^2 dx.$$

If the crudest approximations $h(x) \equiv const$ are considered, the best result is obtained at $h(x) \equiv f_0$; then $\delta(f, f_0) = 1$. Hence, good approximations are the ones with $\delta \ll 1$.

Theorem 3. If $f(x)$ is approximated by $h_L(x)$, then

$$\delta(f, h_L) = 1 - \sum_{s=1}^{L} \sum_{i_1 < \dots < i_s} S_{i_1, \dots, i_s}.$$

Proof. The difference $f(x) - h_L(x)$ is squared and integrated:

$$\int \left[f(x) - h_L(x) \right]^2 dx = \sum_{s=L+1}^{n} \sum_{i_1 < \dots < i_s} D_{i_1, \dots, i_s}.$$

The result is divided by D and the relations (4) and (5) are used.

8 Fixing Unessential Variables

The approximations $h_L(x)$ of the preceding section were low dimensional but the number n of variables remained unchanged. Here we consider the case when several of the input variables have little influence on the output. A common practice is to fix somehow these unessential variables. Let y be the set of important variables and z the set of complementary ones. The set z can be called unessential if $S_z^{t_0 t} \ll 1$.

Let z_0 be an arbitrary value of z in the $(n-m)$-dimensional unit hypercube. As an approximation for $f(x) \equiv f(y,z)$ the function $h = f(y,z_0)$ can be suggested. The approximation error $\delta(f,h)$ depends on z_0 and shall be written as $\delta(z_0) \equiv \delta(f,h)$. The following theorem shows that $\delta(z_0)$ is of the order of $S_z^{t_0 t}$.

Theorem 4. For an arbitrary z_0

$$\delta(z_0) \geq S_z^{t_0 t},$$

but if z_0 is random and uniformly distributed, then for an arbitrary $\varepsilon > 0$ with probability exceeding $1-\varepsilon$

$$\delta(z_0) < \left(1 + \frac{1}{\varepsilon}\right) S_z^{t_0 t}.$$

The proof of Theorem 4 can be found in I.M. Sobol' (1990). Here we shall only mention a corollary for $\varepsilon = 1/2$:

$$\mathbf{P}\left\{\delta(z_0) < 3S_z^{t_0 t}\right\} \geq 0.5.$$

The very first problem solved with the aid of global sensitivity indices was a technical one. The model depended on 35 variables, and it was defined by a computer code. The designers assumed that 12 of these variables were unessential. They were satisfied when the global sensitivity approach produced the result $S_z^{t_0 t} = 0.02$, here z is a subset of unessential variables).

9 Improved Computation Schemes

A. Saltelli showed that in problems in which several sensitivity indices are computed simultaneously, the algorithm of Section 6 can be improved A. Saltelli (2002). The main idea of improvement looks very innocently: both values $f(x)$ and $f(x')$ should be used.

Theorem 1 can be applied to the subset z. Then

$$D_z = \int f(\mathrm{x}')f(y,z')\,\mathrm{d}y\,\mathrm{d}x' - f_0^2$$

and

$$D_y^{tot} = D - D_z,$$

therefore both indices S_y and S_y^{tot} can be computed using three values: $f(x)$, $f(x')$ and $f(y,z')$. Only one of these values depends on the choice of the set $\mathbf{y}$, while the computational algorithm described in Section 6 included two such values, namely (y,z') and $f(y',z)$.

Consider the problem of estimating all one-dimensional indices S_i and S_i^{tot}, $\mathbf{1} \leq i \leq n$. A Monte Carlo algorithm similar to the one presented in Section 6 which would require $n+2$ model evaluations for each trial: $f(x)$, $f(x')$ and $f(x'_1,\ldots,x'_{i-1},x'_i,x'_{i+1},\ldots,x'_n)$, $1 \leq i \leq n$ can be formulated (a direct use of the algorithm from Section 6 would require $2n+1$ model evaluations.)

Moreover, these $n+2$ model evaluations can be used for computing all two-dimensional indices S_{ij}. Indeed, let y_{ij} be the set (x_i, x_j). It follows from Theorem 1 that

$$D_{y_{ij}} = \int f(\ldots,x_i,\ldots)f(\ldots,x_j,\ldots)\,\mathrm{d}x'\,\mathrm{d}x_i\,\mathrm{d}x_j - f_0^2.$$

Here the omitted variables are the coordinates of x'. The two arguments of f differ in two positions only, namely x_i and x_j.

Further, $D_{ij} = D_{y_{ij}} - D_i - D_j$ and $S_{ij} = D_{ij}/D$.

For more details see A. Saltelli (2002).

10 Remarks on the Case of Random Input Variables

Assume that $x_1,\ldots,x_n$ are independent random variables with distribution functions $F_1(x_1),\ldots,F_n(x_n)$, and $f(x_1,\ldots,x_n)$ is a random variable with a finite variance

$$D = Var(f).$$

The definition (1) of ANOVA-decomposition remains true but requirements (2) and (3) should be replaced by corresponding expectations:

$$f_0 = \mathbf{E}f(x)$$

and

$$\int_{-\infty}^{\infty} f_{i_1,\ldots,i_s}\,dF_{i_p}(x_{i_p}) = 0 \quad \text{for} \quad 1 < p < s.$$

In this case the variances $D_{i_1,\ldots,i_s}$ are real variances:

$$D_{i_1,\ldots,i_s} = Var(f_{i_1,\ldots,i_s}).$$

Functional relations that include random variables are true with probability 1.

In F. Campolongo and A. Rossi (2002) it is shown that uncertainty and sensitivity analysis can be valuable tools in financial applications. A delta hedging strategy is analyzed. Considered in the paper financial instrument to be hedged is a caplet, which is an interest rate sensitive derivative. The instrument chosen to hedge the caplet is a forward rate agreement (FRA). The hedging error is defined as the discrepancy between the value of the portfolio at maturity and what it would have been gained investing the initial value of the portfolio at the risk free rate till maturity.

The delta hedging error is considered as a random variable with a certain distribution centered on zero and the target objective function is the 5^{th} percentile of this distribution (VaR). A Monte Carlo experiment is performed in order to obtain the hedging error empirical distribution and to estimate its 5^{th} percentile. Uncertainty analysis is then used to quantify

the uncertainty in the variable of interest, while sensitivity analysis is used to identify where this uncertainty is coming from, which is what factors are causing the value of the maximum loss to be uncertain.

There are seven factors contributing to the uncertainty in this value. These include: the features of the caplet (resetting time, interest rate agreed at the outset of the contract, tenor), the parameters of the model (the mean reverting parameter and the spot rate volatility), the strategy used to build the hedging portfolio (represented as a trigger factor describing the type of movements in the yield curve with respect to which the portfolio is immunized), and the number of times at which the portfolio is updated.

Results for the first order indices showed that nearly 55% of the output variance was due to interaction effects among factors. For models with such a high nonadditivity, the total indices represent a more meaningful measure to look at. Analysis of the total indices showed that almost all input factors were similarly important on the output. The hedging trigger factor was the most important one. As expected, the caplet resetting time was the less important factor.

11 Example: Brownian Bridge

A) *Problem*. Consider a Wiener path integral

$$I = \int_C F[x(t)]\, d_w x,$$

where C is the set of functions $x(t)$ continuous in the interval $0 \le t \le T$ with an initial condition $x(0) = 0$ The integral can be interpreted as an expectation

$$I = \mathbf{E}F[\xi(t)],$$

where $\xi(t)$ is a random Wiener process or Brownian motion which starts with $\xi(0) = 0$. In practical computations $\xi(t)$ is approximated by random polygonal functions $\xi_n(t)$ and expectations

$$I_n = \mathbf{E}F[\xi_n(t)]$$

are estimated by crude Monte Carlo estimators

$$I_{n,N} = \frac{1}{N}\sum_{k=1}^{N} F[\xi_{n,k}(t)]$$

that stochastically converge: $I_{n,N} \xrightarrow{P} I_n$; here $\xi_{n,k}(t)$ are independent realizations of $\xi_n(t)$.

In Yu.A. Shreider (1996), two algorithms for constructing $\xi_n(t)$ were described. In both algorithms the time interval $0 \le t \le T$ is divided into n equal parts and random values of the process $\xi(t)$ at moments $t = \frac{i}{n}T$ are sampled, $1 \le i \le n$. Each value $\xi\left(\frac{i}{n}T\right)$ requires one random normal variate ζ (with parameters 0;1). Then adjacent points $\left(\frac{i}{n}T, \xi(\frac{i}{n}T)\right)$ in the (t, x), plane are connected by straight lines and thus polygonal line $\xi_n(t)$ is constructed.

In the first algorithm which is often called Standard the random values are sampled in the natural order:

$$\xi\left(\frac{1}{n}T\right), \xi\left(\frac{2}{n}T\right), \ldots, \xi\,(T).$$

In the second algorithm it is assumed that n is an integer power of 2, and conditional distributions for the middle of a time interval are applied. The order of sampling is

$$\xi(T), \xi\left(\frac{1}{2}T\right), \xi\left(\frac{1}{4}T\right), \xi\left(\frac{3}{4}T\right), \xi\left(\frac{1}{8}T\right), \ldots, \xi\left(\frac{n-1}{n}T\right).$$

The second algorithm became later known as the Brownian bridge P. Jaeckel (2002).

The probability distributions for $\xi_n(t)$ in both algorithms are the same, hence the variances of $F[\xi_n(t)]$ are equal, and the corresponding Monte Carlo estimators are equivalent. However, it was known that in quasi-Monte Carlo implementations the Brownian bridge is superior to the Standard algorithm. References can be found in I.M. Sobol' and S.S. Kucherenko (2004), where this conclusion was confirmed by sensitivity analysis.

B) *Model and its analysis*. As a model functional we consider the functional from Yu.A. Shreider 1996:

$$F[x(t)] = \int_0^T x^2(t)\, dt.$$

Assume that $T = 1$, and the diffusion coefficient in the definition of Wiener's measure is 0.5. Then $I = \frac{1}{2}$ and the variance $Var\,(F[\xi(t)]) = \frac{1}{3}$.

For both algorithms the integral

$$F_n = \int_0^1 \xi_n^2(t)\, dt$$

can be computed analytically and the result is

$$F_n = \sum_i a_i \zeta_i^2 + \sum_{i<j} a_{ij}\zeta_i\zeta_j,$$

where $\zeta_1, \ldots, \zeta_n$ are independent values of ζ. The coefficients a_i and a_{ij} are different for both algorithms despite the fact that the expectation

$$I_n = \mathbf{E}F_n = \sum_i a_i$$

and the variance

$$Var(F_n) = 2\sum_i a_i^2 + \sum_{i<j} a_{ij}^2$$

are the same. For example, the coefficients a_i at $n = 4$ are

$$a_1 = \frac{10}{48}, \quad a_2 = \frac{7}{48}, \quad a_3 = \frac{4}{48}, \quad a_4 = \frac{1}{48}$$

for the Standard algorithm and

$$a_1 = \frac{16}{48}, \quad a_2 = \frac{4}{48}, \quad a_3 = \frac{1}{48}, \quad a_4 = \frac{1}{48}.$$

for the Brownian bridge.

The ANOVA-decomposition of F_n is

$$F_n = I_n + \sum_i a_i \left(\zeta_i^2 - 1\right) + \sum_{i<j} a_{ij}\zeta_i\zeta_j.$$

There are one-dimensional and two-dimensional terms only and

$$S_i = \frac{Var\left[a_i(\zeta_i^2 - 1)\right]}{Var(F_n)} = \frac{2a_i^2}{Var(F_n)}.$$

Table 1 from I.M. Sobol' and S.S. Kucherenko (2004) contains sums of one-dimensional sensitivity indices at different n for both algorithms as well as values of I_n and variances $Var(F_n)$.

TABLE 1

n	I_n	$Var(F_n)$	$\sum S_i$ Stand	$\sum S_i$ BB
4	0.452	0.323	0.4367	0.7207
8	0.479	0.331	0.2361	0.7214
16	0.489	0.332	0.1222	0.7215
32	0.495	0.333	0.0612	0.7215

Clearly the main contribution to F_n in the Brownian bridge comes from one-dimensional terms(approximately 72%), while for the Standard algorithm the role of two-dimensional terms increases with n. As a rule, in quasi-Monte Carlo, one-dimensional integrals are evaluated with greater accuracy than integrals of higher dimensions. Therefore, the Brownian bridge is more accurate than the Standard algorithm.

C) *Numerical example*. In I.M. Sobol' and S.S. Kucherenko (2004), the integral I_n at $n = 64$ was estimated. Fig.1 shows integration errors ε versus N. To reduce the scatter in error values, the errors were averaged over $K = 50$ independent runs:

$$\varepsilon = \left\{ \frac{1}{K} \sum_{p=1}^{K} \left(I_{n,N}^p - I_n\right)^2 \right\}^{\frac{1}{2}}.$$

For Monte Carlo computations different pseudo-random numbers were used for each run. For quasi-Monte Carlo computations nonoverlapping sections of the *Sobol sequence* were used.

In full agreement with the discussion above, in Monte Carlo both algorithms produce similar errors. However, in quasi-Monte Carlo the errors of the Brownian bridge are much lower.

From the last five points of each line convergence rates were estimated. They were $\sim 1/\sqrt{N}$ for both Monte Carlo lines and $\sim 1/N$ for both quasi-Monte Carlo lines.

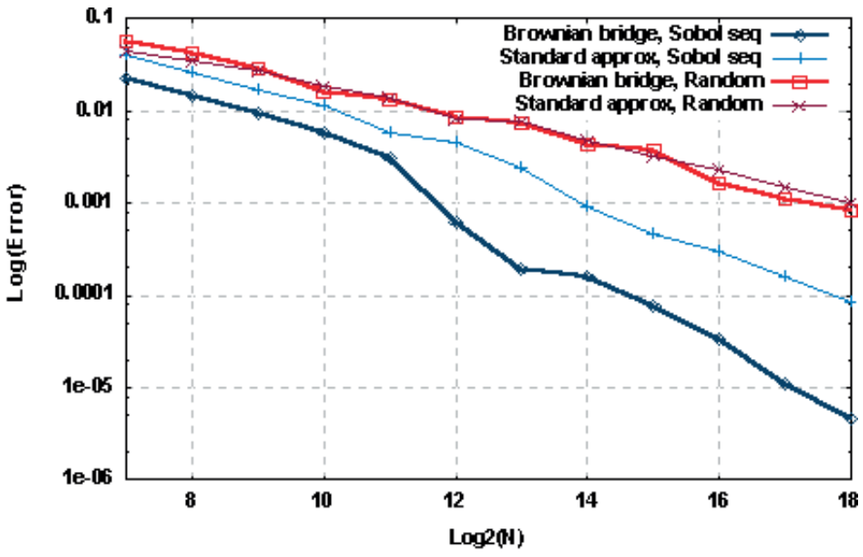

Figure 1: ε **versus** N **at** $n = 64$

Final Remark

In our example, the sensitivity indices for F_n were evaluated analytically. In general, Monte Carlo or quasi-Monte Carlo computations should be used. To avoid a loss of accuracy when f_0 is large, use $f(x) - c$ rather than $f(x)$, with an arbitrary $c \approx f_0$.

REFERENCES

- F. Campolongo, A. Rossi. Sensitivity analysis and the delta hedging problem. In: *Proceedings of PSAM6, 6th International Conference on Probabilistic Safety Assessment and Management*, Puerto Rico, June 23–28, 2002.
- P. Jaeckel. *Monte Carlo Methods in Finance*. J. Wiley, London, 2002.
- H. Rabitz, O.F. Alis, J. Shorter, K. Shim. Efficient input-output model representation. *Computer Physics Communications*, 1999, Vol,117, N 1–2, 11–20.
- A. Saltelli, K. Chan, M. Scott(eds). *Sensitivity Analysis*, London, J.Wiley, 2000.
- A. Saltelli. Making best use of model evaluations to compute sensitivity indices. *Computer Physics Communications* 2002, V.145, 280–297.
- Yu. A. Shreider(ed), *The Monte Carlo Method*, London, Pergamon Press, 1966.
- I.M. Sobol'. Sensitivity estimates for nonlinear mathematical models. *Matematicheskoe Modelirovanie*, 1990, V. 2, N 1, 112–118 (in Russian), English translation in: *Mathematical Modeling and Computational Experiment*, 1993, V. 1, N 4, 407-414.
- I.M. Sobol'. Global sensitivity indices for nonlinear mathematical models and their Monte Carlo estimates. *Mathematics and Computers in Simulation*, 2001, V. 55, N 1–3, 271–280.
- I.M. Sobol'. Theorems and examples on high dimensional model representations, *Reliability Engineering and System Safety*, 2003, Vol,79, 187–193.
- I.M. Sobol', S.S. Kucherenko. On global sensitivity analysis of quasi-Monte Carlo algorithms, 2004 (submitted to Monte Carlo Methods and Applications).